Derivative Analytics and Collateral

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ABSTRACT

This article addresses the valuation of collateralized derivatives that is a critical component of the plumbing of financial system. We present new analytics that models collateral process directly based on bankruptcy laws and Credit Support Annex. The model is very useful for pricing outstanding collateralized derivatives. Empirically we gauge the impact of collateral agreement on derivative valuation and risk management. Our findings indicate that there are important interactions between market and credit risk.

Key words: collateral management, derivative pricing, plumbing of financial system, credit value adjustment, market risk

Collateral arrangements are regulated by the Credit Support Annex (CSA) and is a critical component of the plumbing of financial system. The use of collateral in financial markets has increased sharply over the past decade, yet the effect of collateralization on valuation and risk is an understudied area.

We present a model that characterizes a collateral process directly based on the fundamental principal and legal structure of CSA. The model is devised that allows for collateralization adhering to bankruptcy laws. As such, it can back out price changes due to counterparty risk and collateral posting. Our model is very useful for valuing off-the-run or outstanding derivatives.

This article uses an indirect empirical approach. We define a *swap premium spread* as the premium difference between two swap contracts that have exactly the same terms and conditions but are traded with different CSA counterparties. We reasonably believe that if two contracts are identical except counterparties, the swap premium spread should reflect counterparty credit risk only, as all other risks/costs are identical.

Interest rate swaps collectively account for two-thirds of all outstanding derivatives. An ISDA mid-market swap rate is based on a mid-day polling. Dealers use this market rate as a reference and make some adjustments to quote an actual swap rate. The adjustment or swap premium is determined by many factors, such as credit risk, liquidity risk, funding cost, operational cost and expected profit, etc.

Unlike generic mid-market swap rates, swap premia are determined in a competitive market according to the basic principles of supply and demand. A swap client first contacts a number of swap dealers for a quotation and then chooses the most competitive one. If a premium is too low, the dealer may lose money. If a premium is too high, the dealer may lose the competitive advantage.

Empirically, we obtain a unique proprietary dataset from an investment bank. We use these data and a statistical measurement R^2 to examine whether credit risk and collateralization, alone or in combination, are sufficient to explain market swap premium spreads. The estimation result shows that the adjusted R^2 is 0.7472, implying that approximately 75% of market spreads can be explained by counterparty credit risk. In other words, counterparty risk alone can provide a good but not overwhelming prediction on spreads.

We then assess the joint effect. Because implied or model-generated spreads take into account both counterparty risk and collateralization, we assign the model-implied spreads as the explanatory variable and the market spreads as the response variable. The new adjusted R^2 is 0.9906, suggesting that counterparty risk and collateralization together have high explanatory power on premium spreads. The finding leads to practical implications, such as collateralization modeling allows forecasting credit spread.

Second, we select all the CSA counterparty portfolios in the dataset and then compute their CVAs. We find that the CVA of a collateralized counterparty portfolio is always smaller than the one of the same portfolio without collateralization. We also find that credit risk is negatively correlated with collateralization as an increase in collateralization causes a decrease in credit risk. The empirical tests corroborate our theoretical conclusions that collateralization can reduce CVA charges and mitigate counterparty risk.

The rest of this article is organized as follows: First we present a new model for pricing collateralized financial derivatives. Then we discuss empirical evidences. Finally, the conclusions and discussion are provided. All proofs and detailed derivations are contained in the appendices.

Pricing Collateralized Derivatives

We consider a filtered probability space (Ω, $\mathcal{F}$, $\{\mathcal{F}_t\}_{t\geq 0}$, $\mathcal{P}$) satisfying the usual conditions, where Ω denotes a sample space, $\mathcal{F}$ denotes a σ-algebra, $\mathcal{P}$ denotes a probability measure, and $\{\mathcal{F}_t\}_{t\geq 0}$ denotes a filtration. In the reduced-form framework, the stopping or default time τ of a firm is modeled as a Cox arrival process whose first jump occurs at default and is defined by,

$$\tau = \inf\left\{t : \int_0^t h(s,\Gamma_s)ds \geq \Delta\right\} \tag{1}$$

where $h(t)$ or $h(t,\Gamma_t)$ denotes the stochastic hazard rate dependent on an exogenous common state Γ_t, and Δ is a unit exponential random variable independent of Γ_t.

It is well-known that the survival probability from time *t* to *s* in this framework is defined by

$$p(t,s) := P(\tau > s \mid \tau > t) = \exp\left(-\int_t^s h(u)du\right) \tag{2a}$$

The default probability for the period (*t*, *s*) is given by

$$q(t,s) := P(\tau \le s \mid \tau > t) = 1 - p(t,s) = 1 - \exp\left(-\int_t^s h(u)du\right) \tag{2b}$$

Let valuation date be *t*. Consider a financial contract that promises to pay a $X_T > 0$ at maturity date $T > t$, and nothing before the time.

The binomial default rule considers only two possible states: default or survival. For a discrete one-payment period (*t*, T) economy, at time T the contract either defaults with the default probability $q(t,T)$ or survives with the survival probability $p(t,T)$. The survival payoff is equal to X_T and the default payoff is a fraction of X_T: φX_T, where φ is the recovery rate. The value of this defaultable contract at time *t* is the discounted expectation of all the possible payoffs and is given by

$$V^N(t) = E\left\{D(t,T)\left[p(t,T) + \varphi(T)q(t,T)\right]X_T \middle| \mathcal{F}_t\right\} = E\left[D(t,T)I(t,T)X_T \middle| \mathcal{F}_t\right] \tag{3}$$

where $E\left\{\bullet \middle| \mathcal{F}_t\right\}$ is the expectation conditional on $\mathcal{F}_t$, $D(t,T)$ denotes the risk-free discount factor at time *t* for maturity *T* and $I(t,T) = \left[p(t,T) + \varphi(T)q(t,T)\right]$ can be regarded as a risk-adjusted discount ratio.

Suppose that there is a CSA agreement between a bank and a counterparty in which the counterparty is required to deliver collateral when the mark-to-market (MTM) value arises over the effective threshold *H*.

The default payment under a CSA can be mathematically expressed as

$$P^D(T) = C(T) + \varphi(T)\left(X_T - C(T)\right) = \varphi(T)X_T + C(T)(1 - \varphi(T)) \tag{4}$$

where $C(T)$ is the collateral amount at *T*.

According to CSA, if the contract value $V^C(t)$ is less than the effective threshold $H(t)$, no collateral is posted; otherwise, the required collateral is equal to the difference between the contract value and the effective threshold. The collateral amount posted at time *t* can be expressed mathematically as

$$C(t) = \max(V^C(t) - H(t), 0) \tag{5}$$

where $H(t) = 0$ corresponds to full-collateralization; $H(t) > 0$ represents partial-collateralization; and $H(t) < 0$ reflects over-collateralization.

The value of the CSA contract is the discounted expectation of all the payoffs and is given by

$$V^C(t) = E\left\{D(t,T)\left[q(t,T)\left(C(T) + \varphi(T)(X_T - C(T))\right) + p(t,T)X_T\right] \middle| \mathcal{F}_t\right\} \tag{6}$$

After some simple mathematics, we have the following proposition

Proposition 1*: The value of a collateralized single-payment contract is given by*

$$V^C(t) = E\left[F(t,T)X_T \middle| \mathcal{F}_t\right] - G(t,T) \tag{7a}$$

where

$$F(t,T) = \left(1_{V^N(t) \le H(t)} + 1_{V^N(t) > H(t)} / \bar{I}(t,T)\right) I(t,T) D(t,T) \tag{7b}$$

$$G(t,T) = 1_{V^N(t) > H(t)} H(t) \bar{q}(t,T)\left(1 - \varphi(T)\right) / \bar{I}(t,T) \tag{7c}$$

where $\bar{I}(t,T) = E\left(I(t,T) \middle| \mathcal{F}_t\right)$. $I(t,T)$ and $V^N(t)$ are defined in (3).

Proof: See the Appendix.

The pricing in Proposition 1 is relatively straightforward. We first compute $V^N(t)$ and then test whether its value is greater than $H(t)$. After that, the calculations of $F(t,T)$, $G(t,T)$ and $V^C(t)$ are easily obtained.

We discuss a special case where $H(t) = 0$ corresponding to full-collateralization. Suppose that default probabilities are uncorrelated with interest rates and payoffs[1]. From Proposition 1, we can easily

[1] Moody's Investor's Service [2000] presents statistics that suggest that the correlations between interest rates, default probabilities and recovery rates are very small and provides a reasonable comfort level for the uncorrelated assumption.

obtain $V^C(t) = V^F(t)$ where $V^F(t) = E\left[D(t,T)X_T \,|\mathcal{F}_t\right]$ is the risk-free value. That is to say: the value of a fully-collateralized contract is equal to the risk-free value.

Proposition 1 can be easily extended from single payment to multiple payments. Suppose that a defaultable contract has *m* cash flows represented as X_i with payment dates T_i, where *i* = 1,…,*m*. We derive the following proposition:

Proposition 2: *The value of a collateralized multiple-payment contract is given by*

$$V^C(t) = \sum_{i=1}^{m} E\left[\prod_{j=0}^{i-1}\left(F(T_j,T_{j+1})\right)X_i \,|\mathcal{F}_t\right] - \sum_{i=0}^{m-1} E\left[\prod_{j=0}^{i-1}\left(F(T_j,T_{j+1})\right)G(T_i,T_{i+1})\,|\mathcal{F}_t\right] \tag{8a}$$

where

$$F(T_j,T_{j+1}) = \left(1_{J(T_j,T_{j+1})\le H(T_j)} + 1_{J(T_j,T_{j+1})>H(T_j)} / \bar{I}(T_j,T_{j+1})\right)I(T_j,T_{j+1})D(T_j,T_{j+1}) \tag{8b}$$

$$G(T_j,T_{j+1}) = 1_{J(T_j,T_{j+1})>H(T_j)} H(T_j)\bar{q}(T_j,T_{j+1})\left(1-\varphi(T_{j+1})\right)/\bar{I}(T_j,T_{j+1}) \tag{8c}$$

$$J(T_j,T_{j+1}) = E\left[D(T_j,T_{j+1})I(T_j,T_{j+1})\left(V^C(T_{j+1}) + X_{j+1}\right)\Big|\mathcal{F}_{T_j}\right] \tag{8d}$$

where $\prod_{j=0}^{i-1} F(T_j,T_{j+1}) = 1$ is an empty product when $i = 0$. Empty product allows for a much shorter mathematical presentation of many subjects.

The valuation in Proposition 2 has a backward nature. The most popular backward induction algorithms are lattice/tree and regression-based Monte Carlo.

Empirical Results

In this subsection, we study how collateralization affects credit risk by measuring CVA changes due to collateral posting. CVA is the market price of counterparty credit risk that has become a central part of counterparty credit risk management.

We find that there are a total of 3052 counterparties having live trades as of May 11, 2012. 516 of them have CSA agreements. We randomly select one counterparty portfolio that contains 476 interest rate swaps, 36 interest rate swaptions and 223 interest rate caps/floors. First, we compute the risk-free value

$V^F = 2{,}737{,}702$ that is relatively straightforward as the risk-free portfolio value is what trading systems or pricing models normally report.

We assume that there is counterparty credit risk but no collateral agreement. Based on the pricing model proposed by Xiao [2015], we compute the risky value of the portfolio as $V^N = 2{,}688{,}014$ after considering counterparty credit risk. By definition, the CVA without collateralization is equal to $CVA_N = V^F - V^N = 49{,}688.$

Next, we further assume that there is a CSA agreement in which the threshold is 2 million and the MTA is 100,000. The risky value of the portfolio is calculated as $V^C = 2{,}725{,}094$ according to Proposition 2. The CVA with collateralization is given by $CVA_C = V^F - V^C = 12{,}608$. Similarly, we can compute the CVAs under different collateral arrangements and present the results in Exhibit 9.

Exhibit 9. Impact of Collateralization on CVA

This exhibit shows that CVA increases with collateral threshold. The infinite collateral threshold is equivalent to no collateral agreement and the zero-value collateral threshold corresponds to full collateralization. An increase in collateral threshold leads to a decrease in collateralization.

Effective Threshold	0	2.1 Million	4.1 Million	6.1 Million	8.1 Mil	Infinite (∞)
CVA	0	12,608	23,685	33,504	42,254	49,688

Exhibit 9 tells us that collateral posting can reduce CVA. Full collateralization makes a portfolio appear to be risk-free. An increase in collateral threshold leads to a rise in unsecured credit exposure, and thereby an increase in CVA. In particular, CVA reaches the maximum when the threshold is infinite representing no collateral arrangement.

We extend our analysis to other CSA portfolios. The results hold across different CSA counterparties and collateral agreements. Our findings show that collateral posting can reduce credit risk and CVA. The results also suggest a negative correlation between collateralization and CVA as an increase

in collateralization causes a decrease in CVA charge and vice-versa. These findings improve our understanding of the relationship between collateralization and CVA.

Conclusion

This article presents a new model for pricing collateralized financial contracts based on the fundamental principal and legal structure of CSA. The model can back out market prices. This is very useful for pricing outstanding collateralized derivatives.

Empirically, we use a unique proprietary dataset to measure the effect of collateralization on pricing and compare it with model-implied prices. The empirical results show that the model-implied prices are quite close to the market-quoted prices, suggesting that the model is fairly accurate on pricing collateralized derivatives.

We find strong evidence that counterparty credit risk alone plays a significant but not overwhelming role in determining credit-related spreads. Only the joint effect of collateralization and credit risk has high explanatory power on unsecured credit costs. This finding suggests that failure to properly account for collateralization may result in significant mispricing of derivatives.

We also find evidence that there is a strong linkage between market and credit risk. Our research results suggest that banks and regulators need to think about an integrated framework to capture material interactions of these two types of risk. This requires all profits and losses are gauged in a consistent way across risk types as they tend to be driven by the same economic factors.

Appendix

Proof of Proposition 1: The binomial default rule considers only two possible states: default or survival. For a discrete one-payment period (t, T) economy, at time *T*, the contract either defaults with the default probability $q(t,T)$ or survives with the survival probability $p(t,T)$. The survival value is the

promised payoff X_T and the default payment is $C(T)+\varphi(T)\big(X_T - C(T)\big)$. The value of this collateralized contract is the discounted expectation of all the payoffs and is given by

$$V^C(t) = E\Big\{D(t,T)\big[q(t,T)\big(C(T)+\varphi(T)(X_T - C(T))\big) + p(t,T)X_T\big]\,\big|\,\mathcal{F}_t\Big\} \tag{A1}$$

The collateral posted at time t is defined in (5). The value of the collateral at time T becomes $C(T) = C(t)/D(t,T)$, where we consider the time value of money only for collateral assets.

If $V^C(t) > H(t)$, we have $C(t) = V^C(t) - H(t)$ and

$$V^C(t) = V^N(t)/\bar{I}(t,T) - H(t)\bar{q}(t,T)\big(1-\varphi(T)\big)/\bar{I}(t,T) \tag{A2}$$

where $I(t,T) = D(t,T)\big[p(t,T)+\varphi(T)q(t,T)\big]$, $\bar{I}(t,T) = E\big(I(t,T)\big|\mathcal{F}_t\big)$, and $V^N(t) = E\big[I(t,T)X_T\big|\mathcal{F}_t\big]$.

In this case, $V^C(t) > H(t)$ is equivalent to $V^N(t) > H(t)$.

If $V^C(t) \le H(t)$, we have $C(t) = 0$ and

$$V^C(t) = V^N(t) \le H(t) \tag{A3}$$

Combining the two cases of $V^C(t) > H(t)$ and $V^C(t) \le H(t)$, we get

$$V^C(t) = 1_{V^N(t)\le H(t)}V^N(t) + 1_{V^N(t)>H(t)}\Big[V^N(t)/\bar{I}(t,T) - H(t)\bar{q}(t,T)\big(1-\varphi(T)\big)/\bar{I}(t,T)\Big] \tag{A4}$$

or

$$V^C(t) = E\big[F(t,T)X_T\big|\mathcal{F}_t\big] - G(t,T) \tag{A5a}$$

where

$$F(t,T) = \Big(1_{V^N(t)\le H(t)} + 1_{V^N(t)>H(t)}/\bar{I}(t,T)\Big)I(t,T)D(t,T) \tag{A5b}$$

$$G(t,T) = 1_{V^N(t)>H(t)}H(t)\bar{q}(t,T)\big(1-\varphi(T)\big)/\bar{I}(t,T) \tag{A5c}$$

Proof of Proposition 2: Let $t = T_0$. On the first payment day T_1, let $V^C(T_1)$ denote the CSA value of the contract excluding the current cash flow X_1. According to Proposition 1, the CSA value of the contract at t is given by

$$V^C(t) = E\Big[F(T_0,T_1)\big(X_1 + V^C(T_1)\big)\big|\mathcal{F}_t\Big] - G(T_0,T_1) \tag{A6}$$

Similarly, we have

$$V^C(T_1) = E\left[F(T_1,T_2)\left(X_2 + V^C(T_2)\right)\middle|\mathcal{F}_{T_1}\right] - G(T_1,T_2) \tag{A7}$$

Note that $F(T_0,T_1)$ and $G(T_0,T_1)$ are $\mathcal{F}_{T_1}$-measurable. According to the *taking out what is known* and *tower* properties of conditional expectation, we have

$$\begin{aligned} V^C(t) &= E\left[F(T_0,T_1)\left(X_1 + V^C(T_1)\right)\middle|\mathcal{F}_t\right] - G(T_0,T_1) \\ &= \sum_{i=1}^{2} E\left[\prod_{j=0}^{i-1}\left(F(T_j,T_{j+1})\right)X_i\middle|\mathcal{F}_t\right] + E\left[\prod_{j=0}^{1}\left(F(T_j,T_{j+1})\right)V^C(T_2)\middle|\mathcal{F}_t\right] \\ &\quad - \sum_{i=0}^{1} E\left[\prod_{j=0}^{i-1}\left(F(T_j,T_{j+1})\right)G(T_i,T_{i+1})\middle|\mathcal{F}_t\right] \end{aligned} \tag{A8}$$

By recursively deriving from T_2 forward over T_m, where $V^C(T_m) = X_m$, we obtain

$$V^C(t) = \sum_{i=1}^{m} E\left[\prod_{j=0}^{i-1}\left(F(T_j,T_{j+1})\right)X_i\middle|\mathcal{F}_t\right] - \sum_{i=0}^{m-1} E\left[\prod_{j=0}^{i-1}\left(F(T_j,T_{j+1})\right)G(T_i,T_{i+1})\middle|\mathcal{F}_t\right] \tag{A9}$$

where $\prod_{j=0}^{i-1} F(T_j,T_{j+1}) = 1$ is an empty product when $i = 0$.

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